Asymptotic Solution for Combined Free and Forced Convection in Vertical and Horizontal Conduits with Uniform Suction and Blowing

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Significant changes in heat and momentum transfer rates can be caused by finite interfacial velocities and external field forces. This study considers the nonlinear problem of combined free and forced convection in vertical and horizontal two-dimensional conduits with finite transverse velocity.

Similarity transformations for the temperature function have been found which reduce the energy, momentum, continuity, and state equations for these configurations to nonlinear systems of two coupled ordinary differential equations. These equations are solved by approximate methods to give Nusselt numbers and friction factors as well as velocity and temperature profiles.

Stream to wall temperature differences increase with suction and decrease with injection. The effects of interfacial velocity on temperature profiles and heat transfer increase dramatically with Prandtl number. Because blowing and suction strongly affect temperature profiles they substantially modify natural convection effects in vertical flows. Suction creates steeper transverse temperature gradients and accentuates buoyancy effects in vertical flows. Conversely injection flattens temperature profiles and thus decreases the influence of buoyancy on the velocity field.

The purpose of this study is to examine the influence of finite interfacial velocities and gravitational fields on momentum and energy transport in some nonlinear fully developed laminar conduit flows.

A number of practical applications occur, wherein heat transfer from flowing fluids to solid boundaries must be inhibited so that surface temperatures do not exceed design limits. This has led to numerous investigations to find appropriate techniques for reducing heat transfer rates in forced convection systems. One well-known method found to be useful for this purpose employs a secondary flow directed into the stream and normal to the surface. This can be accomplished either by having the surface sublime or by forcing fluid through slots or pores in the solid phase.

Such systems suggest many interesting theoretical and experimental problems. First it is necessary to study the influence of normal velocities at surfaces on the velocity field in the main flow. Then the energy and diffusion equations can be investigated. However in general the momentum, energy, and diffusion equations are coupled and cannot be solved independently. Transport problems of this kind involving laminar boundary layers have been studied rather extensively, but corresponding problems for conduit flows do not seem to be understood as thoroughly. Perhaps this is due to the lack of a suitable similarity transformation for the temperature function in bounded systems. Here simple transformations will be introduced which are applicable to the fully developed region of simple geometric systems.

The fluid mechanical problem of forced flows with finite interfacial velocity v_w has been examined for parallel plate and tubular geometries. Berman (*1*) first introduced an appropriate similarity transformation for the stream function which reduced the two-dimensional Navier-Stokes equations to a single nonlinear total differential equation which he solved, to the first order, by perturbation. Basically Berman's analysis recognizes the linear increase in volumetric flow rate, and therefore in the stream function, because of constant mass addition per unit length created by a uniform interfacial velocity. Subsequently similar ideas were used to study variations on this theme. Some of these studies reflect interest in physical effects and others in mathematical methods for solving ordinary nonlinear differential equations.

Donoughe (*6*) studied semiporous conduits, whereas Yuan and Finkelstein (*32*) considered the porous tube. Each of these investigators employed perturbation methods and expanded the dimensionless stream function in a Taylor series in which the Reynolds number N_{Re_w} based on wall velocity was the expansion parameter. Results obtained in this way are useful only for small interfacial velocities. Consequently it was natural for other investigators to seek solutions valid for large interfacial velocities. Thus Sellars (*29*) and later Yuan (*34*) used the reciprocal of N_{Re_w} as an expansion parameter and obtained asymptotic results for large N_{Re_w}. In this way the problem was bracketed regarding the influence of N_{Re_w}.

Clearly the intermediate range of N_{Re_w} remained to be filled in. Morduchow (*18*) used an approximation which he called the *method of averages* and showed that for injection his results agreed with perturbation analyses for both large and small N_{Re_w}. This technique is appealing since it is simple and results are obtained in closed form, but it is limited to small suction rates. The approximation is actually a combination of collocation and subdomain techniques which belong to a general class of methods that have been discussed and illustrated by Crandall (*4*). Also exact calculations (*2, 7*) corroborate Morduchow's approximate solution very well over the entire range of blowing that he investigated. White et al. (35) have found an exact series solution to the problem which converges rather slowly for large N_{Re_w}.

The effects of blowing and suction on heat transfer in flows between parallel plates and in tubes also have been investigated (21, 28, 33). Both studies of parallel plate systems considered the linear problem of constant cross-flow velocity with no net addition of material to the primary flow. Eckert (8) has discussed some related problems in a very straightforward way. Yuan and Finkelstein's discussion of the constant wall temperature tube flow with linearly increasing bulk velocity employs a first-order perturbation solution for the velocity distribution and therefore is valid only for very small N_{Re_w}. With this distribution the energy equation was reduced to a Sturm-Liouville system, and the necessary eigenvalues and eigenfunctions were calculated. Although it seems that it has not been done, one can generalize Yuan and Finkelstein's heat transfer results to arbitrarily large injection rates by using either Morduchow's or exact velocity distributions. This would be a straightforward calculation if body forces and viscous dissipation are neglected.

Another class of problems involves the combined effects of forced and free convection wherein cross flow is usually neglected. Several authors (10, 12, 20) presented solutions for fully developed temperature and velocity profiles in vertical tubes. A more complete analysis has been presented by Rosen and Hanratty (13), wherein inertia terms were retained and viscosity variations as well as buoyancy were considered. This work discusses earlier calculations in some detail.

Ostrach (22, 23, 24, 25, 26) solved similar problems for vertical parallel plates and included viscous heating and heat source effects. His papers describe systems with both constant and linearly varying wall temperature and employ the method of successive approximations which has been examined in some detail by Ostrounov (27). Furthermore Maslen (17) critically discussed and extended Ostrach's analyses. Some experimental work (11, 16, 30) has been published which seems to verify the theoretical calculations.

Important differences exist between natural convection in vertical and horizontal flows, since in horizontal conduits circulation can occur in planes normal to the direction of flow. Furthermore in all horizontal flows with monotonically increasing or decreasing axial temperatures, axial density variations cause natural convection. In vertical systems buoyancy effects primarily depend on density differences in the transverse direction.

Gill and del Casal (9) obtained an exact solution for horizontal flow between parallel plates with linearly varying wall temperature which included the effects of uniform heat sources and viscous dissipation. The corresponding solution for a tube was given by Morton (19) and refined by del Casal and Gill (5); both analyses for tubes used perturbation methods with series expansions in the buoyancy parameter.

In view of previous work reported it now seems desirable to

1. Obtain heat transfer information for nonlinear bounded flows that is valid for a wide range of N_{Re_w}.
2. Examine the combined influences of buoyancy and finite N_{Re_w} on the fluid mechanical and heat transfer properties of some fully developed forced flows in conduits.
3. Determine the magnitude of these effects for the two limiting cases of vertical and horizontal flow.

This study considers these three questions. First simple transformations for the temperature functions are given which reduce the two-dimensional partial differential equations for vertical or horizontal plates and vertical tubes to consistent systems of coupled ordinary differential equations. These equations are valid for all values of the parameters so long as laminar flow is stable. However results are asymptotic since they apply only to fully developed thermal and hydrodynamic regions.*

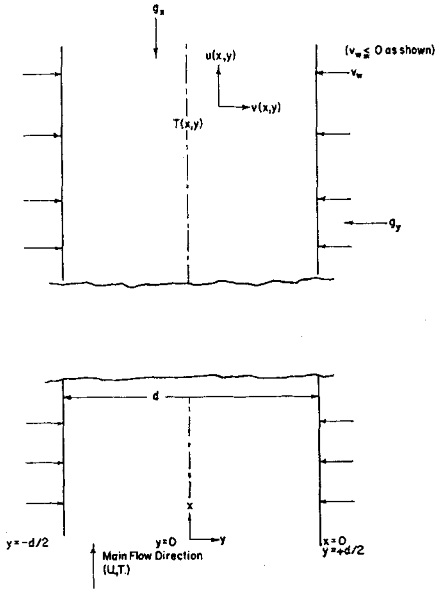

Fig. 1. The system.

DESCRIPTION OF THE SYSTEM

The system analyzed is illustrated in Figure 1, where the velocities in the x and y directions $u(x, y)$ and $v(x, y)$ and the temperature $T(x, y)$ are shown. Gravitational field components are denoted by g_x and g_y. The main stream enters at some axial location $x = 0$ with bulk velocity and temperature U_o and T_o respectively; it is considered to be a two-dimensional laminar flow. The walls, separated by a distance d, are two infinite parallel porous plates through which heat and mass are transferred.

Material crossing the boundaries is assumed to be identical to that in the main stream and either enters or leaves through both walls at identical and constant rates. This causes the mean velocity to vary linearly with x. Both walls are at temperature $T_w(x)$, and material leaving them is assumed to be at wall temperature so that without buoyancy the flow is symmetrical.

The region studied is at large distance from the conduit inlet where velocity and temperature profiles are fully developed. For the vertical orientation ($g_x = g$ and $g_y = 0$) flow is symmetrical about $y = 0$ making it necessary to consider only the profiles in the interval $0 \geqq y \geqq d/2$. In the horizontal case ($g_x = 0$ and $g_y = g$) natural convection causes the flow to be unsymmetrical, so that complete profiles must be determined.

The porous tube problem is quite similar, except the analysis given is limited to vertical flows only. Also for obvious reasons cylindrical coordinates are used in this case.

Fluid properties are constant except for the variation of the density in the buoyancy term. This is a common assumption and allows one to study the influence of N_{Re_w} and field forces without complicating matters by admitting other physical property variations.

* The thermal entrance region is discussed briefly in the Appendix.

ANALYSIS

Here the governing equations are the momentum equations

$$u_1 \frac{\partial u_1}{\partial x_1} + 2v_1 \frac{\partial u_1}{\partial \eta} = -\frac{1}{\rho_o U_o^2} \frac{\partial P}{\partial x_1} + \frac{1}{N_{Re}} \left(\frac{\partial^2 u_1}{\partial x_1^2} + 4 \frac{\partial^2 u_1}{\partial \eta^2} \right) - \frac{dg_x}{U_o^2} \rho/\rho_o \quad (1)$$

$$u_1 \frac{\partial v_1}{\partial x_1} + 2v_1 \frac{\partial v_1}{\partial \eta} = -\frac{2}{\rho_o U_o^2} \frac{\partial P}{\partial \eta} + \frac{1}{N_{Re}} \left(\frac{\partial^2 v_1}{\partial x_1^2} + 4 \frac{\partial^2 v_1}{\partial \eta^2} \right) - \frac{dg_y}{U_o^2} \rho/\rho_o \quad (2)$$

continuity equation

$$\frac{\partial u_1}{\partial x_1} + 2 \frac{\partial v_1}{\partial \eta} = 0 \quad (3)$$

energy equation

$$u_1 \frac{\partial T}{\partial x_1} + 2v_1 \frac{\partial T}{\partial \eta} = \frac{1}{N_{Pr} N_{Re}} \left(\frac{\partial^2 T}{\partial x_1^2} + 4 \frac{\partial^2 T}{\partial \eta^2} \right) + \frac{4\mu U_o^2}{K N_{Re} N_{Pr}} \left(\frac{\partial u_1}{\partial \eta} \right)^2 \quad (4)$$

and the linear equation of state

$$\frac{\rho}{\rho_o} = 1 - \beta (T - T_o) \quad (5)$$

This state equation has been used extensively for both liquids and gases, but it is inadequate near critical conditions (*14, 15*).

To obtain a stream function for the fully developed region one can follow Berman and assume similar velocity profiles, or

$$u_1 (x_1, \eta) / \overline{U}(x_1) = \phi_1(\eta) \quad (6)$$

and a mass balance gives

$$\overline{U} = 1 - 2v_{1w} x_1 \quad (7)$$

Define ψ as usual by

$$u_1 = \frac{\partial \psi}{\partial \eta}, \; v_1 = -\frac{1}{2} \frac{\partial \psi}{\partial x_1} \quad (8)$$

Then if one integrates u_1 with respect to η and uses Equation (6), the result is

$$\psi = \overline{U} f(\eta) \quad (9)$$

This stream function arises quite naturally from physical considerations since the flow rate increases linearly with x_1, and one obtains

$$u_1 = \overline{U} f', \; v_1 = v_{1w} f \quad (10)$$

where prime denotes differentiation with respect to η. Equations (10) are applicable to both vertical or horizontal flows, and the problem is reduced to finding an appropriate temperature function for each attitude.

Vertical Equations and Boundary Conditions

When pressure terms are eliminated from Equations (1) and (2) by differentiating with respect to η and x_1 respectively, and the dimensionless stream function f is introduced using Equations (10), one has

$$\overline{U} \frac{d}{d\eta} \left[f''' - \frac{1}{2} N_{Rew} (ff'' - f'^2) \right] = -\frac{1}{4} N_{Re} \frac{dg}{U_o^2} \frac{\partial T}{\partial \eta} \quad (11)$$

Equation (11) will involve only functions of η if

$$T = T_w + \overline{U} \phi_2 (\eta) \quad (12)$$

where

$$T_w = T_w(0) + A x_1$$

Parameters in these equations may be reduced to a minimum by defining a dimensionless temperature function F as

$$F = \frac{T - T_w}{\frac{1}{4} N_{Re} N_{Pr} \overline{U} \frac{dT_w}{dx_1}} = \frac{T - T_w}{\frac{1}{4} N_{Re} N_{Pr} \overline{U} A} \quad (13)$$

When one neglects viscous dissipation, Equations (11) and (4) then become

$$\frac{d}{d\eta} \left[f''' - \frac{1}{2} N_{Rew} (ff'' - f'^2) \right] = -\frac{1}{16} N_{Ra} F' \quad (14)$$

and

$$f' - \frac{1}{2} N_{Rew} N_{Pr} (f'F - fF') = F'' \quad (15)$$

Clearly Equations (14) and (15) constitute a coupled system of ordinary nonlinear differential equations in η only which completely define the vertical flow problem considered here. Boundary conditions for this case are $f'(1) = 0$ (no slip condition at the wall), $f(1) = 1$ (specified interfacial velocity), $F'(0) = f(0) = f''(0) = 0$ (symmetry), and $F(1) = 0$ (specified wall temperature).

It is interesting that solutions to Equations (14) and (15) are exact particular solutions of Equations (1) through (5), wherein no boundary-layer assumptions are needed. However it was necessary to neglect dissipation effects.

Horizontal Equations and Boundary Conditions

The stream function is the same in this case; hence the equations of motion can be combined by eliminating the pressure and introducing the equation of state to give

$$\overline{U} \frac{d}{d\eta} \left[f''' - \frac{1}{2} N_{Rew} (ff'' - f'^2) \right] = \frac{1}{8} N_{Re} \frac{dg}{U_o^2} \frac{\partial T}{\partial x_1} \quad (16)$$

For similar flows, that is if Equation (16) is independent of x_1, one can specify a temperature function of the form

$$T = T_w + \overline{U}^2 \phi_4(\eta) \quad (17)$$

with

$$T_w = T_w(0) + (x_1 - v_{1w} x_1^2) A \quad (18)$$

Thus

$$\frac{dT_w}{dx_1} = \overline{U} A \quad (19)$$

where A is the dimensionless wall temperature gradient for the case of $v_{1w} = 0$. Therefore define a dimensionless function F_1 by

$$F_1 = \frac{T - T_w}{\frac{1}{4} N_{Re} N_{Pr} \overline{U} \frac{dT_w}{dx_1}} = \frac{T - T_w}{\frac{1}{4} N_{Re} N_{Pr} \overline{U}^2 A} \quad (20)$$

and if axial conduction is neglected, Equations (1), (2), and (4) become

$$\frac{d}{d\eta} \left[f''' - \frac{1}{2} N_{Rew} (ff'' - f'^2) \right] = \frac{1}{8} N - \frac{1}{8} N_{Pr} N_{Rew} N F_1 \quad (21)$$

and

$$f' - \frac{1}{2} N_{Rew} N_{Pr} \left(2f' F_1 - f F_1'\right) =$$

$$F_1'' + \frac{4N_{Br}}{N_{Pr} N_{Re}} (f'')^2 \qquad (22)$$

Boundary conditions for this case are $f'(-1) = f'(1) = 0$ (no slip condition on u_1), $-f(-1) = f(1) = 1$ (specified interfacial velocity), and $F_1(-1) = F_1(1) = 0$ (specified wall temperatures).

Clearly systems of total differential equations for either vertical or horizontal flows are nonlinear, and results obtained by using approximate methods will be presented later. Also notice that the system of coupled equations for vertical flow represents a three-parameter problem. Furthermore the transformation given by Equation (20) applies to the four-parameter problem where viscous dissipation is important, but numerical calculations given later neglect dissipation and emphasize transverse flow and body force effects.

To obtain similar solutions for horizontal flow axial conduction must be neglected while viscous heating may be included, whereas for vertical flow no assumption need be made about axial conduction but viscous heating must be neglected. It is important to note that Equation (22) is valid for any conduit orientation if body force terms are neglected, and thus it enables one to study dissipation effects in a variety of nonlinear bounded flows of practical interest.

Vertical Tubes

With methods used similar to those described previously, in cylindrical coordinates the equations for laminar fully developed combined forced and free convection flow in a porous tube with uniform blowing or suction are found to be

$$\frac{d}{d\eta_t}\left[(\eta_t f_t'')' - N_{Rewt}\left(f_t f_t'' - f_t'^2\right)\right] = -R_t F_t' \qquad (23)$$

$$f_t' - \frac{1}{2} N_{Pr} N_{Rewt} \left(f_t' F_t - f_t F_t'\right) = \frac{1}{2}\frac{d}{d\eta_t}\left(\eta_t F_t'\right) \qquad (24)$$

where

$$F_t = \frac{T - T_w}{\frac{1}{4} N_{Pr} N_{Ret} \overline{U}_t A_t}, \quad \frac{u_1}{\overline{U}_t} =$$

$$2f_t'(\eta_t), \quad \frac{v_1}{v_{1w}} = \frac{1}{\sqrt{\eta_t}} f_t(\eta_t)$$

The boundary conditions are $f_t(0) = 0$ (symmetry), $f_t'(1) = 0$ (no slip at the wall), $f_t(1) = 1/2$ (specified wall velocity), and $F_t(1) = 0$ (specified wall temperature).

Clearly if dissipation is important for tube flow, one can assume $N_{Ra} = 0$ and obtain a result similar to Equation (22).

SOLUTION OF THE RESULTANT EQUATIONS

The technique employed here to solve the vertical flow problem is a combination of the subdomain and collocation methods which have been described by Crandall (4). This approach is referred to as the *method of averages* in Morduchow's work (18). The essential feature of these methods involves the construction of solutions in the form

$$\phi(\eta) = \sum_{i=0}^{K} \phi_i(a_i, \eta)$$

where ϕ_i are functions which satisfy the boundary conditions of the problem under consideration. Then the analyst must choose the a_i so that ϕ is a reasonable approximation to the solution of the differential equation involved. Collocation and subdomain methods are weighted average criteria for choosing the a_i; they differ only with regard to the weighting function used in the averaging process. When collocation is used, the assumed functions are forced to satisfy the differential equations at specific points of the interval. In contrast assumed solutions satisfy the equation in the mean over the interval (or some portion of the interval) when the subdomain technique is used. In each case the arbitrary constants a_i are determined from the resultant algebraic equations. Least-squares and Galerkin's method are similar in principle to the present approach.

Since symmetry exists in vertical flows, it is relatively simple to obtain rather accurate solutions. However the asymmetrical nature of horizontal systems substantially increases the computation involved. Thus a simplified averaging method and a first-order perturbation analysis have been used to obtain approximate solutions for horizontal flows. The perturbation method employs a truncated series expansion with N_{Rew} as the expansion parameter. This approximation is considered to be valid for very small N_{Rew} only.

Complete details concerning the methods used to obtain solutions are given elsewhere (3). Therefore only a brief resume of the more important aspects of the procedure will be discussed here. For horizontal conduits assumed velocity and temperature functions are identical to exact solutions in the limit $N_{Rew} = 0$. Polynomials for the vertical case with no cross flow predicts reversed flow at the wall for $N_{Ra} = -507$, while Ostrach's exact solution yields $N_{Ra} = -501$. This is an error of 1.2% under rather extreme conditions. If $N_{Ra} = 0$, the velocity distribution reduces to Morduchow's result which agrees very well with exact numerical solutions.

To the authors' knowledge Equations (15), (22), and (24) have not been derived previously. Consequently no exact solutions are available for comparison, but since this case is important in a variety of practical applications it seemed necessary to bolster confidence in the heat transfer results obtained.

Equation (15) yields the following iteration formula

$$F = \int_0^{\eta} \exp(Q) \int_0^{\eta} \left[1 - \frac{N_{Pr} N_{Rew}}{2} F\right] f' \exp(Q)\, d\eta\, d\eta$$

where

$$Q = \frac{N_{Pr} N_{Rew}}{2} \int_0^{\eta} f d\eta$$

and by using weighted average solutions as first-order approximations second-order results were obtained for $N_{Ra} = 0$, $N_{Pr} = 1$, and $N_{Rew} = 1, -10$. Since the departure from heat transfer rates with $N_{Rew} = 0$ is of primary concern, percentage changes in $F'(1)$ constitute a reasonable criterion for estimating errors.

In the cases considered first- and second-order results differed from 0.1% to 1.5%, whereas the temperature gradient at the wall changed by more than 90% from that at $N_{Rew} = 0$. Consequently in this case weighted average solutions seem quite adequate for engineering purposes. Furthermore one can expect that N_{Rew} affects heat transfer to a very considerable extent, and it is rather surprising that convective heat transfer in porous

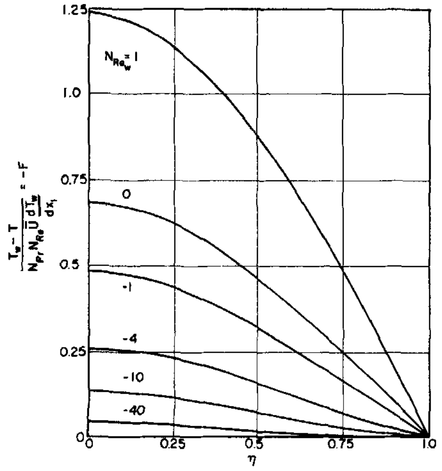

Fig. 2. Temperature profiles for various N_{Re_w} in vertical flows with $N_{Pr} = 1$, $N_{Ra} = -500$.

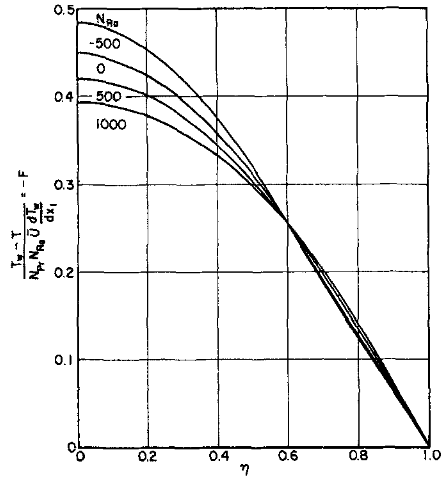

Fig. 3. Temperature profiles for various N_{Ra} in vertical flows with $N_{Pr} = 1$, $N_{Re_w} = -1$.

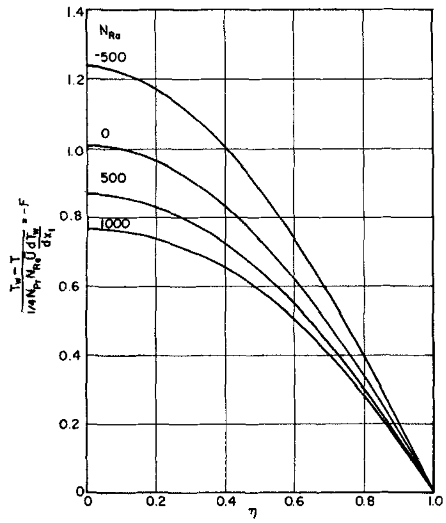

Fig. 4. Temperature profiles for various N_{Ra} in vertical flows with $N_{Pr} = 1$, $N_{Re_w} = 1$.

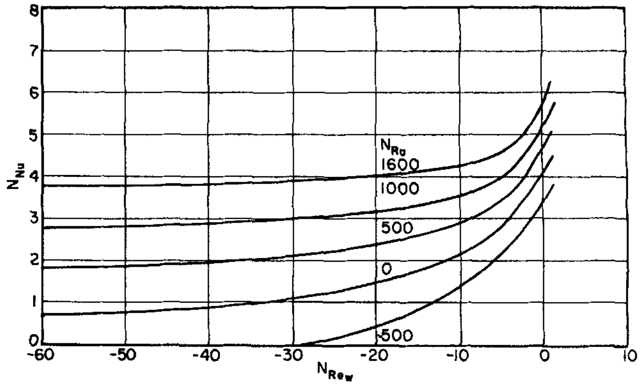

Fig. 5. Nusselt number vs. N_{Re_w} for various N_{Ra} in vertical flows with $N_{Pr} = 1$.

conduit flows has not received more attention particularly because of the potential for temperature control in industrial reactors.

DISCUSSION OF RESULTS

Approximate solutions to the equations for vertical and horizontal flows yield useful information regarding the importance of gravitational field forces and finite interfacial velocity on heat and momentum transfer for forced convection flows in bounded systems. Wherever comparisons can be made, results obtained agree with those of previous studies in which gravitational forces or small interfacial velocities were considered separately. In the following discussion positive and negative N_{Re_w} correspond to suction and injection respectively. N_{Ra} is positive when upflowing fluid is heated or downflows are cooled. Conversely N_{Ra} is negative for upflow cooling or downflow heating.

Vertical Flow Results

Substantial changes in heat transfer rates occur because of interfacial velocities and external force fields. The effect of injection or suction on stream to wall temperature differences is shown in Figure 2 for fixed N_{Pr} and N_{Ra}. Except for extremely low Prandtl number fluids (such as liquid metals $N_{Pr} \sim 0.01$) transverse flows strongly affect temperature profiles and consequently the heat transfer to the wall. Injection markedly decreases heat transfer rates, while suction increases these rates. Another important consequence of injecting material into the stream is the reduction of natural convection in vertical flows. This occurs because injection reduces temperature gradients in the transverse direction which induce buoyancy effects (or natural convection) and modify the velocity field. A comparable change occurs with suction, except stream to wall temperature differences are increased and cause the field force to be more significant. In contrast to velocity distributions temperature profiles are less sensitive to buoyancy as illustrated in Figures 3 and 4. Positive N_{Ra} tends to increase velocities and therefore temperature gradients near the wall as can be seen in Figure 3. This can be made more obvious by referring $T - T_w$ to temperature difference between wall and channel center and plotting this vs. η.

Small injection rates decrease heat transfer markedly, but the relative magnitude of this reduction attenuates as N_{Re_w} increases. This behavior can be seen quite clearly in Figure 5 which also demonstrates that the acceleration of particles near the wall caused by buoyancy when N_{Ra} is positive tends to increase heat transfer rates. Nusselt numbers used are defined by

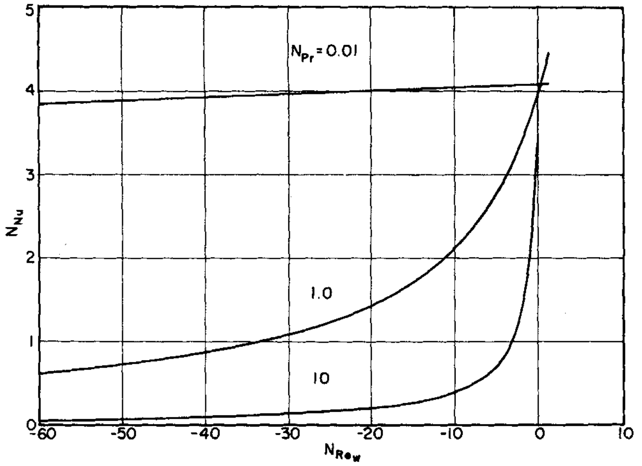

Fig. 6. The effect of N_{Pr} on heat transfer as a function of N_{Re_w} when $N_{Ra} = 0$.

$$N_{Nu} = \frac{hd}{K} = -2 \frac{F'(1)}{\int_0^1 f'Fd\eta} \tag{25}$$

where the heat transfer coefficient h is defined by

$$K \frac{\partial T}{\partial y} \bigg]_{y=d/2} = h\,(T_w - T_b)$$
$$T_b = \text{bulk temperature}$$

Heat transfer rates depend dramatically on N_{Pr} as shown in Figure 6, where injection causes a minor reduction in heat transfer at small Prandtl numbers, which correspond with liquid metals; however injection effects are much more pronounced when N_{Pr} is larger. This occurs primarily because radial convection produces larger effects in fluids characterized by lower molecular conductivity as can be seen by inspecting Equation (15). Roughly $N_{Pr} = 1$ characterizes gases, and the Prandtl number for water is on the order of 10. Since $N_{Ra} = 0$, these results apply to any conduit orientation as long as T_w is linear in x. This is very important, since field forces play a minor role in many forced convection problems. Other studies (*21, 28, 33*) also have shown that heat transfer rates decrease as a result of injection. Using a first-order perturbation velocity distribution and assuming constant wall temperature Yuan and Finkelstein (*33*) determined that heat transfer rates (or N_{Nu}) depend linearly on N_{Re_w} for small injection rates. However they suggested, presumably on intuitive grounds, that the dependence of N_{Nu} on N_{Re_w} should become less pronounced for larger N_{Re_w}. Present results corroborate this statement.

Important fluid mechanical effects are caused by both buoyancy and interfacial velocity in vertical flow. Body forces can be responsible for instabilities in flows with or without interfacial velocity. Theoretical studies (*11, 13, 20, 22, 24, 25*) predict instabilities at the wall when field forces oppose the flow and at midchannel when field forces aid the flow. Experimental study of combined forced and free convection has shown rather clearly that these instabilities do occur (*12, 13, 30, 31*). Figures 7 and 8 illustrate the influence of natural convection on the velocity distribution. With injection as seen in Figure 7 buoyancy effects are less pronounced, whereas Figure 8 shows that larger field force effects occur when N_{Re_w} is

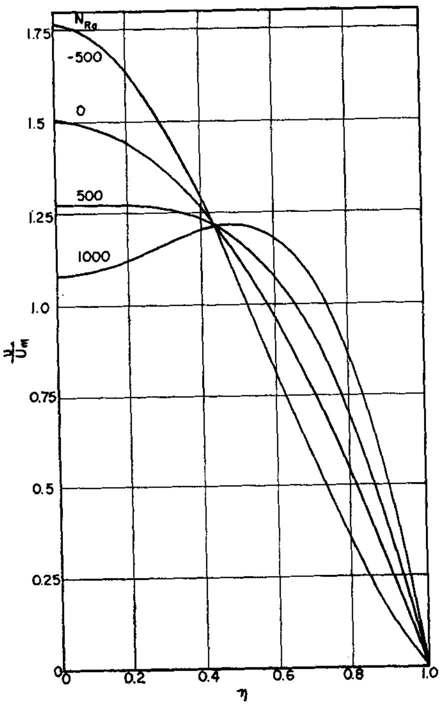

Fig. 7. Velocity profiles for various N_{Ra} in vertical flows and $N_{Pr} = 1$, $N_{Re_w} = 1$.

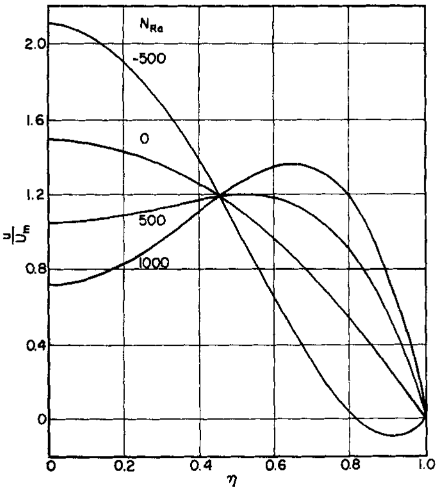

Fig. 8. Velocity profiles for various N_{Ra} in vertical flows and $N_{Pr} = 1$, $N_{Re_w} = 1$.

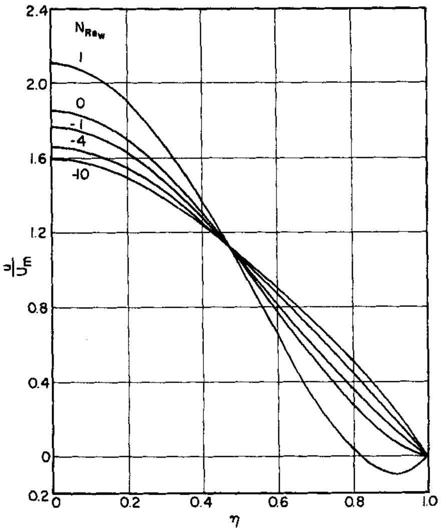

Fig. 9. Velocity profiles for various interfacial velocities in vertical flows with $N_{Pr} = 1$, $N_{Ra} = -500$.

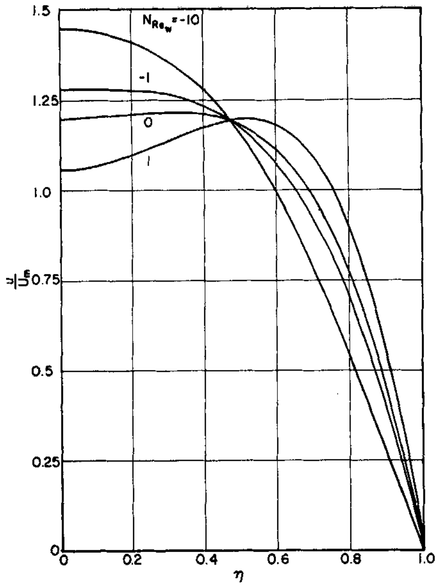

Fig. 10. Velocity profiles for various N_{Re_w} in vertical flows with $N_{Ra} = 500$, $N_{Pr} = 1$.

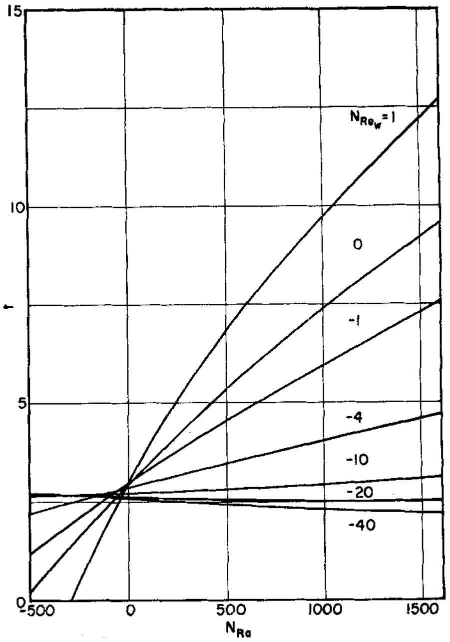

Fig. 12. Friction factor vs. Rayleigh number for various N_{Re_w} in vertical flows with $N_{Pr} = 1$.

positive. N_{Re_w} affects the velocity profiles given in Figure 9, where it can be seen that buoyancy tends to cause reversed flow at the wall. Instabilities also tend to occur in the central portion of the conduit as illustrated in Figures 10 and 11. In all plots field force effects, which tend to accelerate the flow near the wall for positive N_{Ra} and decelerate it for negative N_{Ra}, are decreased by injection and increased by suction.

The dimensionless friction factor t is defined as

$$t = \frac{C_f N_{Re_x}}{4} = -f''(1) \tag{26}$$

where

$$C_f = \frac{-\mu \left.\frac{\partial u}{\partial y}\right]_{y=d/2}}{\frac{1}{2}\rho_o U_m^2}$$

and

$$N_{Re_x} = \overline{U} N_{Re} = \frac{dU_m}{\nu}$$

Dependence of this parameter on both transverse velocity and natural convection is shown in Figure 12. It is apparent that both buoyancy and interfacial velocity effects diminish with increasing injection rates. Figure 13 shows that for smaller N_{Pr} the effect of blowing and suction on t is reduced.

Dimensionless axial pressure gradient π_o is defined as

$$\pi_o = \frac{N_{Re_x}}{\rho_o U_m^2} \left.\frac{\partial (P - P_s)}{\partial x_1}\right]_{y=0} \tag{27}$$

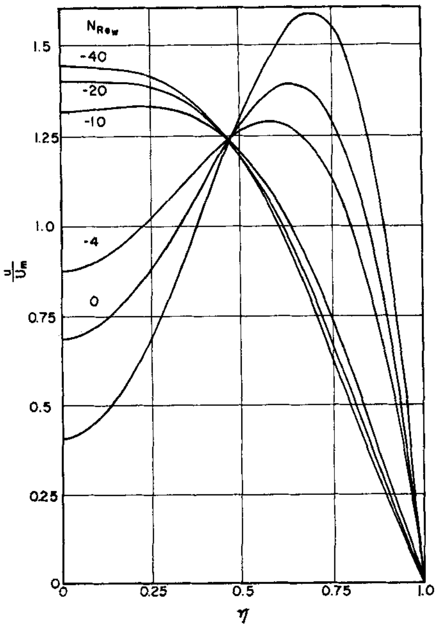

Fig. 11. Velocity profiles for various N_{Re_w} in vertical flows with $N_{Pr} = 1$, $N_{Ra} = 1{,}600$.

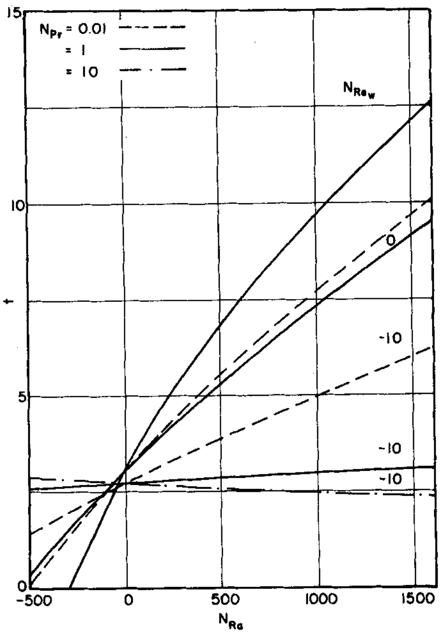

Fig. 13. Friction factor vs. Rayleigh number for various N_{Re_w} and N_{Pr} in vertical flows.

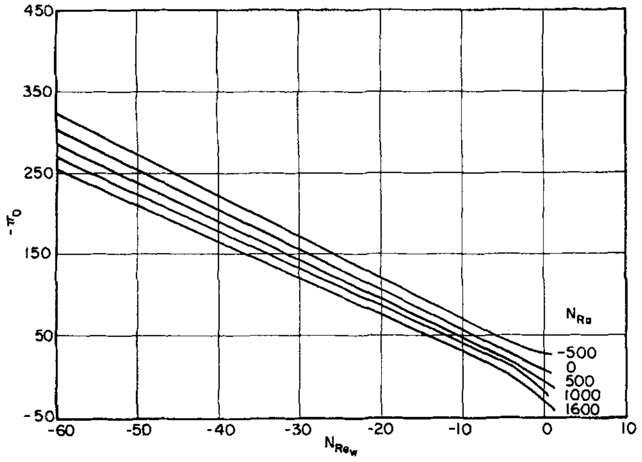

Fig. 14. Pressure parameter vs. injection rate for various Rayleigh numbers in vertical flows with $N_{Pr} = 1$.

where

$$\frac{\partial P_s}{\partial x_1} = -\rho\, gd$$

As one might expect Figure 14 demonstrates that adding mass increases the pressure gradient needed to maintain flow.

In many applications material will be injected into the wall at a different temperature than the desired wall temperature. The temperature of injected fluid $T_c(x)$ needed to maintain a given wall temperature $T_w(x)$ with a stream bulk temperature $T_b(x)$ is found by making a heat balance on the wall to be

$$\frac{T_c - T_w}{T_b - T_w} = \frac{N_{Nu}}{N_{Pr}N_{Re_w}} \tag{28}$$

For nonlinear T_w Equation (28) can be obtained by neglecting the net axial conduction in the wall.

Horizontal Results

Some information concerning the importance of natural convection and interfacial velocity in heat and momentum transport has been obtained for this case from two approximate methods.

Similar to vertical flows stream to wall temperature differences are decreased by injection and increased by suction. However since large changes in heat transfer are created by small N_{Re_w}, the first-order perturbation seems inadequate except when N_{Re_w} is very small. The method of averages results appear in Figure 15. Both methods predict that heat transfer rates will be diminished by injection, but perturbation tends to overestimate this effect. Because the heat transfer to the walls is not perfectly symmetrical Nusselt numbers must be defined for each wall as

$$(N_{Nu})\text{ top plate} = -4\frac{F'(1)}{\int_{-1}^{1} f'F d\eta} \tag{29}$$

$$(N_{Nu})\text{ bottom plate} = 4\frac{F'(-1)}{\int_{-1}^{1} f'F d\eta} \tag{30}$$

When $N_{Ra} = 0$, conduit orientation is immaterial, and therefore heat transfer results for linear and parabolic wall temperature distributions can be compared directly. Qualitatively the variation of N_{Nu} with N_{Re_w} is identical in both cases, and furthermore at $N_{Pr} = 1$, N_{Nu} values for

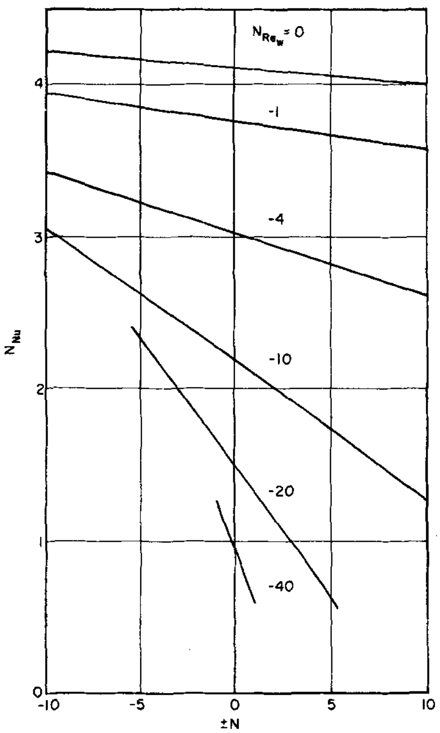

Fig. 15. Nusselt numbers vs. buoyancy parameter for various N_{Re_w} in horizontal flows with $N_{Pr} = 1$. N is positive for top plate and negative for bottom plate.

these wall temperature distributions differed by less than 7% up to $N_{Re_w} = -40$. When $N_{Pr} = 0.01$, the agreement is better; however with larger N_{Pr} somewhat greater differences might be expected. Nevertheless this indicates that heat transfer results determined from the specialized transformations used here, together with those obtained in previous studies, provide a reasonable basis for making design estimates for a rather wide variety of conditions.

Skin friction at the two plates differs because of the asymmetry of the flow; therefore the friction factor is defined for each wall as

$$(t)\text{ top plate} = \frac{(C_f)_{\text{top}} N_{Re_x}}{4} = -f''(1) \tag{31}$$

$$(t)\text{ bottom plate} = \frac{(C_f)_{\text{bottom}} N_{Re_x}}{4} = f''(-1) \tag{32}$$

where

$$(C_f)\text{ top} = \frac{-\mu \frac{\partial u}{\partial y}\Big]_{y=d/2}}{\frac{1}{2}\rho_o U_m^2}$$

$$(C_f)\text{ bottom} = \frac{\mu \frac{\partial u}{\partial y}\Big]_{y=d/2}}{\frac{1}{2}\rho_o U_m^2}$$

Figure 16 indicates how t depends on buoyancy and interfacial velocity.

CONCLUSIONS

1. The partial differential equations for laminar fully developed combined forced and free convection flow between porous parallel plates are reduced to a set of non-

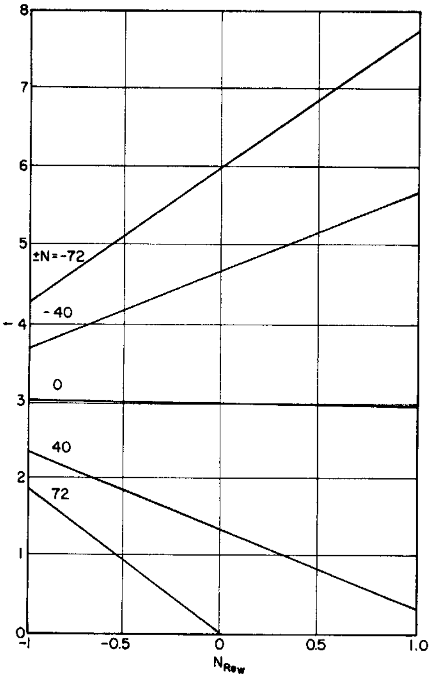

Fig. 16. Friction factor vs. injection rate for various buoyancy parameters in horizontal flows with $N_{Pr} = 1$. N is positive for top plate and negative for bottom plate.

linear total differential equations by using the stream function of Berman (*1*) and the temperature transformation given in this work.

2. Transverse temperature differences increase with suction and decrease with injection.

3. For fluids with Prandtl numbers on the order of 1 or greater significant decreases in heat transfer rates can be obtained by injection.

4. Injection cooling is ineffective for small Prandtl number fluids such as liquid metals ($N_{Pr} \sim 0.01$).

5. Injection can stabilize velocity profiles for cases where natural convection causes instabilities.

6. In contrast suction increases these instabilities.

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NOTATION

A = wall temperature gradient based on x_1
A_t = wall temperature gradient based on x_2

$$C_f = \text{friction factor} = \frac{-\mu \left.\dfrac{\partial u}{\partial y}\right]_{y=d/2}}{\dfrac{1}{2}\rho\, U_m^2}$$

d = distance between plates
f = dimensionless stream function for parallel plate system
f_t = dimensionless stream function for tube
F, F_1 = dimensionless temperature functions for parallel plate systems
F_t = dimensionless temperature function for tube
g = gravitational constant, 32.2 ft./sec.^{2}
g_x = x component of the gravitational force
g_y = y component of the gravitational force
h = heat transfer coefficient
K = thermal conductivity
N = horizontal buoyancy parameter = N_{Gr}/N_{Re}
N_1 = $N/8$
N_{Br} = Brinkman number = $(\mu u_o^2)/(KA)$
N_{Gr} = Grashof number = $(d^3 g\beta A)/(\nu^2)$
N_{Nu} = Nusselt number = $(hd)/(k)$
N_{Pr} = Prandtl number = ν/α
N_{Ra} = Rayleigh number = $(d^3 g\beta A)/(\nu\alpha)$
N_{Re} = Reynolds number = $(dU_o)/(\nu)$
N_{Ret} = Reynolds number for tubes = $(r_o U_o)/(\nu)$
N_{Rew} = wall Reynolds number = $(dv_w)/(\nu)$
N_{Rewt} = wall Reynolds number for tubes = $(r_o v_w)/(\nu)$
N_{Rex} = mean Reynolds number = $(dU_m)/(\nu)$
P = pressure
r = radial distance from tube axis
r_o = radius of a tube
R = $N_{Ra}/16$
R_t = $(r_o^3 g\beta A_1)/(32\,\nu\alpha)$
t = dimensionless friction factor
T = temperature
T_b = bulk mean temperature at any x
T_c = temperature of coolant entering the wall at any x
T_o = reference temperature
T_w = wall temperature
u = velocity in the x direction
u_1 = dimensionless velocity in the x direction = u/U_o
U_o = inlet mean velocity
U_m = mean velocity at any x
U_{mt} = mean velocity at any x for tube flow
$\overline{U}$ = dimensionless mean velocity = U_m/U_o
$\overline{U_t}$ = dimensionless mean velocity ($U_{mt}/U_o = 1 - 2\,v_{1w}x_2$)
v = transverse velocity
v_w = transverse velocity at $y = d/2$ (wall velocity)
v_1 = dimensionless transverse velocity
v_{1w} = dimensionless wall velocity = v_w/U_o
x = axial coordinate
x_1 = dimensionless axial coordinate = x/d
x_2 = dimensionless axial coordinate for a tube = x/r_o
y = transverse coordinate

Greek Letters

α = thermal diffusivity
β = expansion coefficient in the equation of state
η = dimensionless transverse coordinate $(2y/d)$
η_t = dimensionless transverse coordinate $(r/r_o)^2$
μ = viscosity
ν = kinematic viscosity = μ/ρ
π_o = pressure parameter
ρ = density
ρ_o = reference density
ϕ_1 = functions of η
ψ = stream function

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APPENDIX

Asymptotic solutions to the energy equation must reduce to known results when $N_{Re_w} = 0$. Furthermore since the energy equation is linear, it should be possible by superpositior to construct a more general solution which applies to the thermal entrance region.

Clearly the asymptotic solutions given previously reduce exactly to well-known solutions for linear wall temperature. However it may not be quite so obvious that superposition can be used to describe the thermal entrance region. Therefore a very brief discussion will be given to illustrate how this may be done. It is necessary to neglect field forces; thus the momentum and energy equations are uncoupled.

Assume that an entrance section is attached to a system of porous parallel plates so that the velocity distribution entering the thermal test section is fully developed. Consider axial conduction negligible, and for simplicity neglect viscous dissipation, although this can be included without difficulty. If $x_1 = 0$ is the entrance to the test section, then the wall temperature distribution is the same as that in Equation (12), and the fluid entrance temperature is uniform and equal to a constant. Once the basic problem is solved, the entrance temperature distribution can be generalized to arbitrary $T(0, \eta)$.

Let

$$T = T_w + \overline{U}\,\phi_2(\eta) + G(x_1, \eta)$$

and by allowing the asymptotic functions F or ϕ_2 to account for all inhomogeneities one obtains Equation (15), and

$$\overline{U} f' \frac{\partial G}{\partial x_1} + 2V_{1w} f \frac{\partial G}{\partial \eta} = \frac{4}{N_{Pe}} \frac{\partial^2 G}{\partial \eta^2}$$

together with homogeneous boundary conditions

$$G(x_1, 1) = \frac{\partial G}{\partial \eta}(x_1, 0) = 0$$

These equations represent the flat plate linear wall temperature analogue of the constant wall temperature problem discussed by Yuan and Finkelstein (*33*) and can be reduced immediately to a Sturm-Liouville system by separation of variables. An identical procedure can be used for the tube problem.

Free-Radical Yields in *n*-Alcohols Resulting from Gamma Irradiation

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The physical interaction of radiation with matter results in the formation of chemically reactive species. For organic liquids the precursors of most of the chemical products found as a result of irradiation are predominantly free radicals. The yield or G-value (molecules formed/100 electron volt absorbed) for free-radical formation for various organic compounds is a quantity of fundamental interest since, if the identities of the radicals are known in addition to their yield, the overall chemical effect of radiation may be predicted. Many investigators have reversed the procedure and have predicted G-values from the quantity and nature of the final products. Obviously it is desirable to check such results by direct measurements for free-radical formation, and such measurements have been reported in the literature for various compounds. The method most suitable for liquid-phase reactions utilizes some well-known specific reactions for free radicals such as the interaction with highly reactive solutes termed "scavengers." To be useful such reactions must remove all radicals from the system before they may undergo secondary reactions. The stable free-radical diphenylpicrylhydrazyl (DPPH) has been used for this purpose with, however, widely varying results (*4, 6, 7, 16*). These results are also inconsistent with values re-

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